Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
Zenica, 04.06.2013.

Drugi parcijalni iz predmeta **Analiza III**

1. Izračunati zapreminu tijela ograničenog loptom $x^2+y^2+z^2=a^2$, cilindrom $x^2+y^2=ax$ i ravni $0xy$ koji se nalazi u gornjem poluprostoru.

2. Izračunati krivolinijski integral druge vrste

$$I=\oint_C x\mathrm{d}y+x\mathrm{d}z$$

gdje je C kriva koja nastaje presjekom cilindrične površi $x^2+y^2=2x$ i ravni $z=x$ pozitivno orjentisana ako se posmatra iz tačke $(0;0;1)$.

3. Izračunati

$$I=\iint_{S+}\left(\frac{1}{x}\mathrm{d}y\mathrm{d}z+\frac{1}{y}\mathrm{d}z\mathrm{d}x+\frac{1}{z}\mathrm{d}x\mathrm{d}y\right)$$

gdje je $S+$ spoljašnja strana jedinične sfere (zadatak uraditi bez upotrebe teoreme Gauss-Ostrogradskog - zadatak se i ne može uraditi uz pomoć navedene teoreme zato što ne ispunjavaju sve uslove teoreme).

4. Neka funkcije $g,h:\mathbb{R}^3\longrightarrow\mathbb{R}$ ispinjavaju

$$\Delta g(x,y,z)=0 \quad \text{i} \quad \Delta h(x,y,z)=0$$

gdje je $\Delta=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$ Laplace-ov operator. Za funkciju $f:\mathbb{R}^3\longrightarrow\mathbb{R}$ datu sa

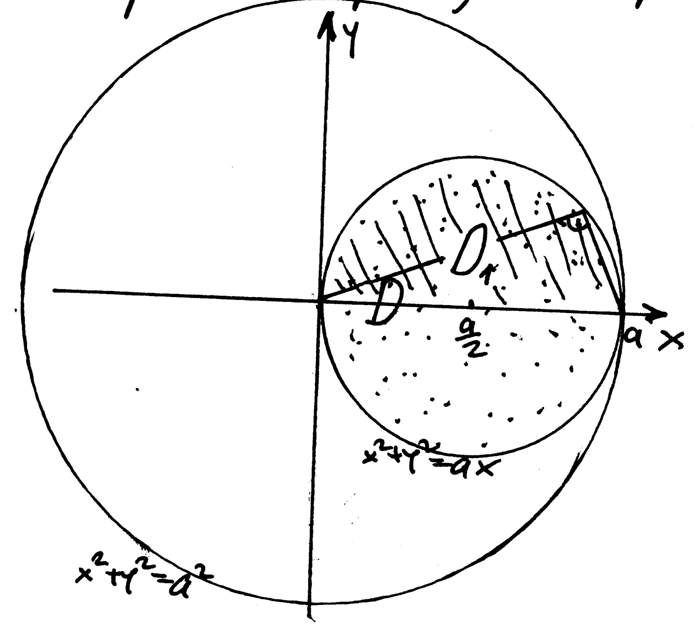
$$f(x,y,z)=g(x,y,z)+(x^2+y^2+z^2)h(x,y,z)$$

izračunati $\Delta\Delta f(x,y,z)$.

Zadaci su skinuti sa stranice pf.unze.ba/nabokov.
Za uočene greške pisati na *infoarrt@gmail.com*

Izračunati zapreminu tijela ograničenog loptom $x^2+y^2+z^2=a^2$, cilindrom $x^2+y^2=ax$ i ravni Oxy koji se nalazi u gornjem poluprostoru.

Rj. Napravimo presjek lopte i cilindra sa ravni xOy.



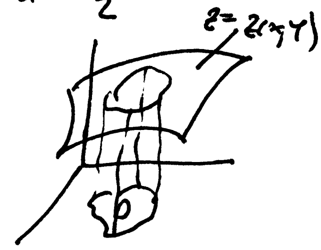
$$x^2+y^2=ax$$

$$x^2-2\cdot x\cdot\frac{a}{2}+\left(\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$

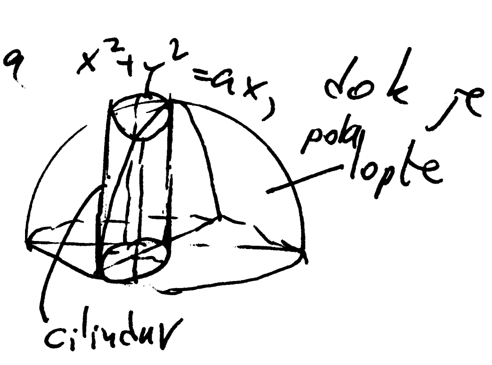
$$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$$

krug sa centrom $C(\frac{a}{2},0)$ poluprečnika $\frac{a}{2}$

$$V=\iint\limits_D z(x,y)\,dxdy$$



U našem slučaju D je unutrašnjost kruga $x^2+y^2=ax$, dok je $z(x,y)$ dio lopte iznad xOy ravni.



$$z^2=a^2-x^2-y^2$$

$$z=\pm\sqrt{a^2-x^2-y^2}\qquad \text{nama treba } +$$

$$V=\iint\limits_D\sqrt{a^2-x^2-y^2}\,dxdy=2\iint\limits_{D_1}\sqrt{a^2-x^2-y^2}\,dxdy=\left|\begin{array}{l}\text{uvedimo cilindrične koordinate}\\ x=\rho\cos\varphi\\ y=\rho\sin\varphi\\ dxdy=\rho\,d\rho\,d\varphi\end{array}\right.\quad D_1\xrightarrow{\text{transf.}}D_1':\Big\{\quad \cos\varphi=\frac{\rho}{a}$$

$$D_1':\begin{cases}0\le\rho\le a\cos\varphi\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}\Bigg|=2\iint\limits_{D_1'}\sqrt{a^2-\rho^2}\,\rho\,d\rho\,d\varphi=\left|\begin{array}{l}d(a^2-\rho^2)=-2\rho\,d\rho\\ \rho\,d\rho=-\frac{1}{2}d(a^2-\rho^2)\end{array}\right|=$$

$$=2\cdot\frac{-1}{2}\int_0^{\pi/2}d\varphi\int_0^{a\cos\varphi}\sqrt{a^2-\rho^2}\,d(a^2-\rho^2)=-\int_0^{\pi/2}\frac{2}{3}(a^2-\rho^2)^{\frac{3}{2}}\Big|_0^{a\cos\varphi}d\varphi=-\frac{2}{3}\int_0^{\pi/2}(\sin^3\varphi-1)\,d\varphi\overset{\text{za vjež.}}{=}\dots=\frac{a^3}{9}(3\pi-4)$$

(#) Izračunati krivolinijski integral druge vrste

$$I = \oint_C x\,dy + x\,dz$$

gdje je C kriva koja nastaje presjekom cilindrične površi $x^2+y^2=2x$ i ravni $z=x$ pozitivno orjentisana ako se posmatra iz tačke $(0;0;1)$.

Rj.

$$C:\begin{cases} x^2+y^2=2x \\ z=x \end{cases}$$

$$x^2+y^2=2x$$
$$x^2-2\cdot x+1+y^2=1$$
$$(x-1)^2+y^2=1$$

krug sa centrom $C(1;0)$ poluprečnika 1

$$x-1=\cos\varphi$$
$$y=\sin\varphi$$

$$\Rightarrow \quad x^2+y^2=2x:\begin{cases} x=1+\cos\varphi \\ y=\sin\varphi \end{cases}$$

Sad nije teško parametrizirati datu krivu C.

$$C:\begin{cases} x=1+\cos\varphi \\ y=\sin\varphi \\ z=1+\cos\varphi \\ 0\le\varphi\le 2\pi \end{cases} \qquad \begin{aligned} dx&=-\sin\varphi\,d\varphi \\ dy&=\cos\varphi\,d\varphi \\ dz&=-\sin\varphi\,d\varphi \end{aligned}$$

$$I=\oint_C x\,dy+x\,dz=\int_0^{2\pi}\Big[(1+\cos\varphi)\cos\varphi+(1+\cos\varphi)(-\sin\varphi)\Big]\,d\varphi=$$

$$=\int_0^{2\pi}\cos\varphi\,d\varphi+\int_0^{2\pi}\cos^2\pi\,d\varphi-\int_0^{2\pi}\sin\varphi\,d\varphi-\int_0^{2\pi}\sin\varphi\cos\varphi\,d\varphi=$$

$$= \underset{\ldots}{\text{ZA VJEŽBU}} = 0+\pi-0-0=\pi$$

traženo rješenje

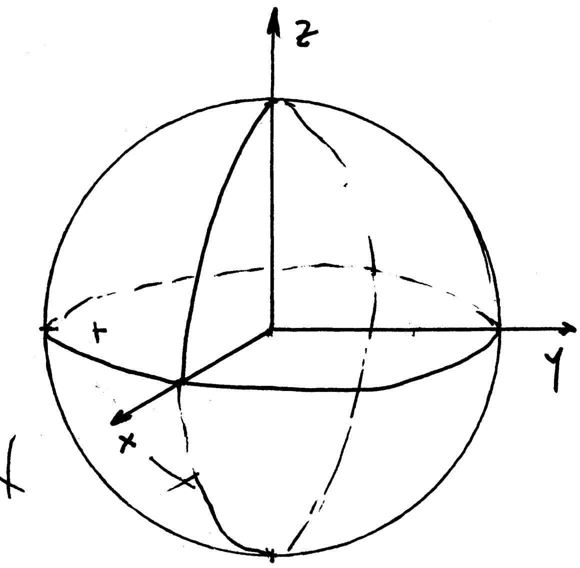
(#) Izračunati $I=\iint\limits_{S+}\left(\frac{1}{x}\,dy\,dz+\frac{1}{y}\,dz\,dx+\frac{1}{z}\,dx\,dy\right)$

gdje je $S+$ spoljašnja strana jedinične sfere (zadatak uraditi bez upotrebe teoreme Gauss-Ostrogradskog – zadatak ne ispunjava sve uslove teoreme)

zato što se ne može uraditi uz pomoć navedene teoreme

Rj. Podjelimo dati integral na tri dijela:

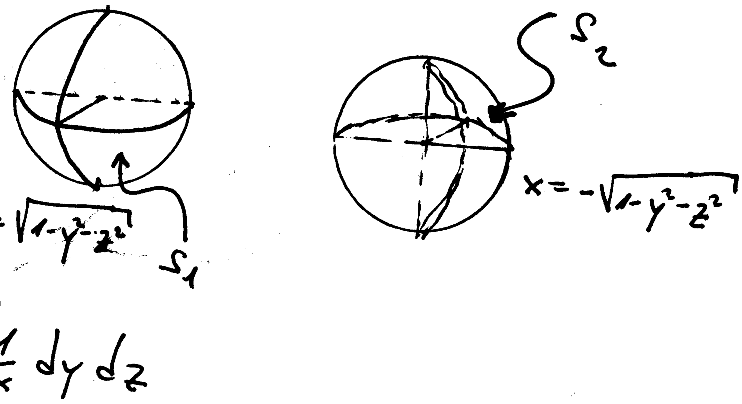
$$I=\iint\limits_{S+}\frac{1}{x}\,dy\,dz+\iint\limits_{S+}\frac{1}{y}\,dz\,dx+\iint\limits_{S+}\frac{1}{z}\,dx\,dy=I_1+I_2+I_3$$



$x^2+y^2+z^2=1$

Posmatrajmo $I_1=\iint\limits_{S+}\frac{1}{x}\,dy\,dz$

Zbog vektora normale, u ovom slučaju, spoljašnju stranu jedinične sfere trebamo podjeliti na dva dijela S_1 i S_2



$$I_1=\iint\limits_{S+}\frac{1}{x}\,dy\,dz=\iint\limits_{S_1}\frac{1}{x}\,dy\,dz+\iint\limits_{S_2}\frac{1}{x}\,dy\,dz$$

$\iint\limits_{S_1}\frac{1}{x}\,dy\,dz=$

- ugao α između vektora normale $\vec{n}$ i x-ose je između 0 i $\pi/2$; $0\le\cos\alpha\le 1$ $\cos\alpha\ge 0$
- $x=\sqrt{1-y^2-z^2}$
- ortogonalna projekcija od S_1 na xOy ravan je krug $y^2+z^2=1$

$$=+\iint\limits_{D}\frac{dy\,dz}{\sqrt{1-y^2-z^2}}=$$

$$= \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ y = \rho\cos\varphi \\ z = \rho\sin\varphi \\ dy\,dz = \rho\, d\rho\, d\varphi \end{array} \quad D \xrightarrow{\text{transf.}} D' : \begin{cases} 0 \le \rho \le 1 \\ 0 \le \varphi \le 2\pi \end{cases} \right| = \iint\limits_{D'} \frac{\rho\, d\rho\, d\varphi}{\sqrt{1-\rho^2}} = \int_0^1 \frac{\rho\, d\rho}{\sqrt{1-\rho^2}} \int_0^{2\pi} d\varphi =$$

$$= \left| \begin{array}{l} d(1-\rho^2) = -2\rho\, d\rho \\ \rho\, d\rho = -\frac{1}{2} d(1-\rho^2) \end{array} \right| = 2\pi \int_0^1 \frac{-\frac{1}{2} d(1-\rho^2)}{\sqrt{1-\rho^2}} = -\pi \cdot \frac{(1-\rho^2)^{\frac{1}{2}}}{\frac{1}{2}} \Bigg|_0^1 =$$

$$= -2\pi(0-1) = 2\pi$$

$$\iint\limits_{S_2} \frac{1}{x}\, dy\, dz = \left| \begin{array}{l} \bullet\ x = -\sqrt{1-y^2-z^2} \\ \bullet\ \alpha \in (\pi/2, \pi) \Rightarrow \cos\alpha \le 0 \\ \bullet\ \text{ortog. proj. je } D: y^2+z^2=1 \end{array} \right| = -\iint\limits_{D} \frac{dy\, dz}{-\sqrt{1-y^2-z^2}} = \iint\limits_{S_1} \frac{1}{x}\, dy\, dz$$

$$= 2\pi$$

Primjetimo da ista priča važi za $I_2 = \iint\limits_{S+} \frac{1}{y}\, dx\, dz$, $I_3 = \iint\limits_{S+} \frac{1}{z}\, dx\, dy$ (u oba slučaja $S+$ se podjeli na dvije polusfere i čitav račun bude sličan računu iznad). Prema tome

$$I = 4\pi + 4\pi + 4\pi = 12\pi \quad \text{traženo rješenje.}$$

(#) Neka f-je $g, h : \mathbb{R}^3 \to \mathbb{R}$ ispunjavaju

$$\Delta g(x,y,z) = 0 \quad i \quad \Delta h(x,y,z) = 0$$

gdje je $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ LAPLACE-ov operator.

Za f-ju $f : \mathbb{R}^3 \to \mathbb{R}$ datu sa

$$f(x,y,z) = g(x,y,z) + (x^2+y^2+z^2)\, h(x,y,z)$$

izračunati $\Delta\Delta f(x,y,z)$.

Rj.

$$\Delta f(x,y,z) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$f(x,y,z) = g(x,y,z) + (x^2+y^2+z^2)\, h(x,y,z)$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x} + 2x\, h + (x^2+y^2+z^2)\frac{\partial h}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial x^2} + 2h + 2x\frac{\partial h}{\partial x} + 2x\frac{\partial h}{\partial x} + (x^2+y^2+z^2)\frac{\partial^2 h}{\partial x^2}$$

$$= \frac{\partial^2 g}{\partial x^2} + 2h + 4x\frac{\partial h}{\partial x} + (x^2+y^2+z^2)\frac{\partial^2 h}{\partial x^2} \qquad \ldots(1)$$

slično bi dobili

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial x^2} + 2h + 4y\frac{\partial h}{\partial y} + (x^2+y^2+z^2)\frac{\partial^2 h}{\partial y^2} \qquad \ldots(2) \quad i$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 g}{\partial z^2} + 2h + 4z\frac{\partial h}{\partial z} + (x^2+y^2+z^2)\frac{\partial^2 h}{\partial z^2} \qquad \ldots(3)$$

Kada saberemo (1), (2) i (3) dobićemo

$$\Delta f = \underbrace{\Delta g}_{=0} + 6h + 4x\frac{\partial h}{\partial x} + 4y\frac{\partial h}{\partial y} + 4z\frac{\partial h}{\partial z} + (x^2+y^2+z^2)\underbrace{\Delta h}_{=0} =$$

$$= 6h(x,y,z) + 4x\frac{\partial}{\partial x}h(x,y,z) + 4y\frac{\partial}{\partial y}h(x,y,z) + 4z\frac{\partial}{\partial z}h(x,y,z)$$

$$\Delta f = 6h + 4x\frac{\partial h}{\partial x} + 4y\frac{\partial h}{\partial y} + 4z\frac{\partial h}{\partial z}$$

$$\Delta\Delta f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Delta f = \frac{\partial^2 \Delta f}{\partial x^2} + \frac{\partial^2 \Delta f}{\partial y^2} + \frac{\partial^2 \Delta f}{\partial z^2}$$

$$\frac{\partial \Delta f}{\partial x} = 6\frac{\partial h}{\partial x} + 4\frac{\partial h}{\partial x} + 4x\frac{\partial^2 h}{\partial x^2} + 4y\frac{\partial^2 h}{\partial x \partial y} + 4z\frac{\partial^2 h}{\partial x \partial z}$$

$$= 10\frac{\partial h}{\partial x} + 4x\frac{\partial^2 h}{\partial x^2} + 4y\frac{\partial^2 h}{\partial x \partial y} + 4z\frac{\partial^2 h}{\partial x \partial z}$$

$$\frac{\partial^2 \Delta f}{\partial x^2} = 10\frac{\partial^2 h}{\partial x^2} + 4\frac{\partial^2 h}{\partial x^2} + 4x\frac{\partial^3 h}{\partial x^3} + 4y\frac{\partial^2 h}{\partial x^2 \partial y} + 4z\frac{\partial^2 h}{\partial x^2 \partial z}$$

$$= 14\frac{\partial^2 h}{\partial x^2} + 4x\frac{\partial^3 h}{\partial x^3} + 4y\frac{\partial^3 h}{\partial x^2 \partial y} + 4z\frac{\partial^3 h}{\partial x^2 \partial z}$$

Slično

$$\frac{\partial^2 \Delta f}{\partial y^2} = 14\frac{\partial^2 h}{\partial y^2} + 4x\frac{\partial^3 h}{\partial x \partial y^2} + 4y\frac{\partial^3 h}{\partial y^3} + 4z\frac{\partial^3 h}{\partial y^2 \partial z}$$

$$\frac{\partial^2 \Delta f}{\partial z^2} = 14\frac{\partial^2 h}{\partial z^2} + 4x\frac{\partial^3 h}{\partial x \partial z^2} + 4y\frac{\partial^3 h}{\partial y \partial z^2} + 4z\frac{\partial^3 h}{\partial z^3}$$

Prema tome

$$\Delta\Delta f = 14\Delta h + 4x\frac{\partial}{\partial x}\Delta h + 4y\frac{\partial}{\partial y}\Delta h + 4z\frac{\partial}{\partial z}\Delta h$$

Kako je $\Delta h = 0$ prema postavci zadatka to je

$$\Delta\Delta f = 0$$